THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5000 Analysis I 2015-2016
Problem Set 8: Towards Topology and Functional Analysis

1. Let $X = \{a, b, c\}$ and let $\mathfrak{I} = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\}$.

   Show that $\mathfrak{I}$ defines a topology on X.

2. Let X be a set and let $\mathfrak{I}$ be the collection of all subsets U of X such that $X \backslash U$ either is finite or is X. Show that $\mathfrak{I}$ defines a topology on X, which is called the **finite complement topology**.

3. Suppose that $\mathcal{B}$ is a basis of a set X. Let $\mathfrak{I}$ be the collection of all subsets U of X such that for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

   Show that $\mathfrak{I}$ defines a topology on X, which is called the **topology generated by** $\mathcal{B}$.

4. Let $\mathcal{B}$ be the collection of all intervals in $\mathbb{R}$ which are in the form

$$(a, b) = \{x \in \mathbb{R} \, : \, a < x < b\}.$$

   Show that $\mathcal{B}$ is a basis of $\mathbb{R}$.

5. Let $\mathcal{B}$ be the collection of all disks in $\mathbb{R}^2$ which are in the form

$$D(\overrightarrow{x_0}, r) = \{\overrightarrow{x} \in \mathbb{R}^2 \, : \, |\overrightarrow{x} - \overrightarrow{x_0}| < r\},$$

   where $r > 0$.

   Show that $\mathcal{B}$ is a basis of $\mathbb{R}^2$.

6. Let $\{x_n\}$ be a convergent sequence in a Hausdorff space X. Show that the limit of the sequence $\{x_n\}$ is unique.

7. Let $X = \{(a_1, a_2, a_3) \, : \, a_1, a_2, a_3 \in \{0, 1\}\}$. Define $d : X \times X \to \mathbb{R}$ be the number of distinct components.

   Show that d defines a metric on X.

8. Let $X = \{a, b, c\}$ and define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 & \text{if} \quad x = y, \\ \\ 1 & \text{if} \quad x \neq y. \end{cases}$$

   (a) Show that d defines a metric on X.

   (b) If $\mathfrak{I}$ is the metric topology induced by the metric d, descibe $\mathfrak{I}$ explicitly.

9. Define $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$ by

$$\|(x_1, x_2, \cdots, x_n)\| = \max\{x_1, x_2, \cdots, x_n\}$$

   Show that $\| \cdot \|$ defines a norm on $\mathbb{R}^n$.

10. Let $\mathcal{R}[a,b]$ be the collection of all (Riemann) integrable functions on $[a,b]$.

Define $\|\cdot\| : \mathcal{R}[a,b] \to \mathbb{R}$ by

$$\|f\| = \sqrt{\int_a^b |f|^2}.$$

Does $\|\cdot\|$ define a norm on $\mathcal{R}[a,b]$?

11. Let $\mathcal{C}[a,b]$ be the collection of all continuous functions on $[a,b]$.

Define $\|\cdot\| : \mathcal{C}[a,b] \to \mathbb{R}$ by

$$\|f\| = \sqrt{\int_a^b |f|^2}.$$

Does $\|\cdot\|$ define a norm on $\mathcal{C}[a,b]$?